Modelling Firms and Markets

Unit 1 Static Games of Complete Information

Contents

Unit Overview

This unit starts the first half of the module (Units 1–4), which provides a solid foundation to game theory and an introduction to the study of firms' behaviour in strategic interactions. Each of these units contains two parts: theory and its applications. This aims to broaden and deepen your understanding in how to apply game theory in diverse areas, such as economics, finance and management. Unit 1 starts with the fundamental concepts of game theory by introducing normal-form games and discussing pure- and mixed-strategy Nash equilibrium in finite games. Then we apply the concept of Nash equilibrium to two typical oligopoly models in the market: the Cournot and Bertrand models.

Learning outcomes

When you have completed the unit and its readings, you will be able to:

- state the reasoning of dominant strategy equilibrium in simple games
- explain the basic equilibrium concepts such as Nash equilibrium (pure and mixed)
- find pure- and mixed-strategy Nash equilibrium in simple games
- discuss the sufficient conditions and intuition of the existence of Nash equilibrium
- solve simple oligopoly games (Cournot and Bertrand).

Reading for Unit 1

Robert Gibbons (1992) *A Primer in Game Theory*. Harlow, UK, Pearson Education. Chapter 1 'Static games of complete information'.

1.1 Normal (Strategic) Form Game and Iterated Deletion

Game theory is a unique and revolutionary part of microeconomics. It adopts ideas from various disciplines (including economics, mathematics, philosophy, psychology and other social and behavioural sciences), and develops into a mathematical application in determining optimal outcomes of conflict and cooperative strategies among reasonably rational agents. It has been applied in many fields for decision-making processes outside academic studies, such as auction formats, political decisions, business strategies, *etc*. For instance, a computer manufacturer may need to decide whether to launch its new computer immediately to gain a competitive edge or to prolong the testing period of its new functions. This kind of decision can be extended into different areas and usually involves a number of parties. Decision makers can use game theory as a tool to map out possible strategies with corresponding results and make rational decisions. I hope that you will enjoy learning about such an interesting decision-making process in the first half of this module.

In general, games are divided into two branches: cooperative and non-cooperative games. This module focuses on the non-cooperative games that mainly examine how players interact with each other in order to achieve their own goals (no binding agreements). You will evaluate some simplified and fundamental examples in the real-world economic and financial environments, such as wages and employment in a unionised firm, auctions, sequential bargaining, *etc*. In Unit 1, you focus on simultaneous-move (so-called *static*) games of complete information. By *complete information* we mean that all aspects of the game structure are common knowledge among all the players. There is no private information, such as each player's payoff function, the timing and other information of the game. I will discuss all of this in more detail later on in this unit.

In any game, the outcome depends on the strategy chosen by each player, which is the key to the whole of game theory. You should bear in mind the following points from now on.

- *Strategy* – a complete contingent plan that specifies an action for every information set (a particular set of possible moves) of the player.
- *Player's decision problem* – the choice of a strategy that a player thinks would counter the best strategies adopted by the other players.

Osborne and Rubinstein (1994) summarise the characteristics of the strategic game as a model of an event that occurs only once:

- each player knows the details of the game and the fact that all players are rational
- each player is unaware of the choices of other players
- each player chooses the strategy simultaneously and independently.

Reading 1.1

Please read Sections 1.1.A and 1.1.B, pp. 2–8 of your key text by Gibbons. This illustrates some basic ideas of the normal form presentation and iterated elimination of strictly dominated strategies.

Gibbons (1992) Section 1.1.A 'Normal-Form Representation of Games' and 1.1.B 'Iterated Elimination of Strictly Dominated Strategies' in *A Primer in Game Theory*. pp. 2–8.

Gibbons has generalised the case and provided a formal definition of a normal-form game. To understand that definition, you need to define three essential elements of a normal-form game:

1. A finite set of players, $I = \{1,\ 2,\ \ldots, n\}$
2. For each player i, a finite set of strategies, S_i, (where $s_i \in S_i$. A strategy, S_i, is a member of the set of strategies, S_i.)
3. For each player i, there is a payoff function,

   $$u_i : S_1 \times S_2 \times \ldots S_I \to R\ ,\ I = \{1,\ 2,\ \ldots, n\}$$

   that associates with each strategy combination $(s_1, s_2 \ldots, s_n)$, a payoff $u_i(s_1, s_2, \ldots, s_n)$ for player i.

In any strategic interaction, it is crucial for players to consider not only what their opponents will do, but also what opponents know, which strategies their opponents will choose, *etc*. Indeed, many games can be simplified through iterated deletion of dominated strategies based on common knowledge and rationality. Gibbons explains the term of common knowledge on page 7 of the key text. In a two-player game, common belief in (or *knowledge of*) rationality means that player 1 believes player 2 is rational, player 2 believes that player 1 believes that player 2 is rational, and player 1 believes that player 2 believes that... Thus, there is a common belief/knowledge among the players if they all know it, all know that they all know it, and so on. Iterated dominance is a method of narrowing down the set of strategies of playing the game. Gibbons gives a formal definition of strictly dominated strategy, which I will summarise here.

Definition 1

The pure strategy s_i is strictly dominated for player i if there exists

$$s_i' \in S_i \quad \text{such that} \quad u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})\ ,\ \forall s_{-i}\ .$$

If a player has a dominated strategy in a game, you need to know that this is the strategy the player will not use. Let us consider a crucial example – the Prisoners' Dilemma. This game requires a single round of elimination of dominated strategies to solve the problem. The scenario is that two prisoners are interrogated and each has two strategies. The payoffs are as follows.

For Prisoner 1, if he/she plays 'Fink', his/her payoff is either 0 or –6, which is higher than the payoff from playing 'Mum'. This is also true for Prisoner 2. Thus, a rational player would never Mum. That is, a prisoner will always choose 'Fink' without even knowing the other prisoner's payoff. Therefore, (Fink, Fink) will be the outcome reached by two rational players.

Figure 1.1

Prisoner 1 \ Prisoner 2	Mum (not confess)	Fink (confess)
Mum (not confess)	−1, −1	−9, 0
Fink (confess)	0, −9	−6, −6

You should be able to discuss the following points:

1. The iterated deletion of strictly dominated strategies solution is the set of all the strategies that survive the indefinite process of iterated deletion of strictly dominated strategies. In some games, no strategies can be eliminated. However, in certain games, all strategies except one for each player can be eliminated, then that game is said to be *dominance-solvable*.
2. The equilibrium outcome, (Fink, Fink), is neither optimal nor efficient, because if players can coordinate – *ie* (Mum, Mum), they would have obtained higher payoffs. Accordingly, (Fink, Fink) is Pareto dominated[1] by (Mum, Mum). The result shows the value of commitment of playing strategy Mum credibly. (This point will be discussed later on in this module.)
3. In a game, if all players are rational and there is common belief of rationality, each player will choose a strategy that survives iterated strong deletion.
4. If multiple strategies are strictly dominated, then they can be eliminated in any sequence without changing the set of strategies that we end up with.

Let us now practise with another example below. The game is according to Figure 1.1.1 of Gibbons, page 6. Make sure you know how to eliminate dominated strategies iteratively. Please note that the order of deletion does not matter.

[1] When an outcome is *Pareto dominated*, it means that all the agents/players prefer other outcomes. In contrast, an outcome is *Pareto optimal* if no other outcomes would be preferred by all the players.

Figure 1.2

Player 1 \ Player 2	Left	Middle	Right
Up	1, 0	1, 2	0, 1
Down	0, 3	0, 1	2, 0

Step 1 For player 1, there is no dominated strategy.
Step 2 Then for player 2, Right is dominated by Middle. Eliminate Right.

Then, the game is reduced to a 2 × 2 game, as shown in Gibbons' Figure 1.1.2:

Figure 1.3

Player 1 \ Player 2	Left	Middle
Up	1, 0	1, 2
Down	0, 3	0, 1

Step 3 In the remaining game, for player 1, Down is dominated by Up.Eliminate Down.
Step 4 Once Down is removed for player 1. For player 2, Left is dominated by Middle.

This gives us (Up, Middle) as the unique equilibrium.

You should now know how iterated elimination of strictly dominated strategies work, but have you noticed two drawbacks of the concept?

- Assume that it is common knowledge that each player is rational. This assumption can be too strong under experiment.
- Many games may not have dominated or weakly dominated strategies. Therefore, the criteria of dominance or weak dominance may not work in some games. We need to look at an alternative concept in the next section – the Nash equilibrium (NE). This concept is more precise – that is, the players' strategies in a Nash equilibrium always survive iterated elimination of strictly dominated strategies, but not the reverse. Furthermore, all finite games have at least one Nash equilibrium. (This may involve mixed-strategy Nash equilibrium which we will discuss in Section 1.3).

1.2 Nash Equilibrium

We introduce the Nash equilibrium in this section with a reading from the key text by Gibbons (1992).

Gibbons (1992) Section 1.1.C 'Motivation and definition of Nash equilibrium' in *A Primer in Game Theory*. pp. 8–12.

Reading 1.2

Please read Gibbons, Chapter 1, Section 1.1.C, 'Motivation and definition of Nash equilibrium', pp. 8–12.

As you read, make notes on the definition and uses of the Nash equilibrium. NE is a fundamental concept, and you should make sure that you are very familiar with its intuition and be able to apply it later on in this unit. To clarify, all the Nash equilibria referred to in 1.1.C of Gibbons (1992) are pure strategy Nash equilibria.

Definition 2

A strategy profile $\left(s_i^*, s_{-i}^*\right)$ is a (pure strategy) Nash equilibrium if for each player *i*,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$$

for all players *i* and all $s_i \in S_i$.

This shows that an NE is any profile in which each player is choosing optimally given the choices of the other players. NE represents a strategically stable situation in which no player anticipates higher payoff from unilateral deviation. That is, a Nash equilibrium is a set of strategies (one for each player), such that no player has incentive to unilaterally change his/her action. For example, in a simple 2 × 2 game, both players know which strategy the other player is going to choose, and no player has an incentive to deviate from the equilibrium strategy because his/her strategy is a best response to his/her belief about the other player's strategy.

Now, let us look at some important examples used by Gibbons. Here, I highlight the crucial points of solving these games. You should try to do so yourself.

Example 1: Prisoners' Dilemma Revisit – Games with Unique NE

Figure 1.1

Prisoner 1 \ Prisoner 2	Mum (not confess)	Fink (confess)
Mum (not confess)	–1, –1	–9, 0
Fink (confess)	0, –9	–6, –6

Look at this game. It has a unique NE (Fink, Fink), because each player can deviate from the other strategy profile profitably. (Mum, Mum) and (Mum, Fink) cannot be NE because prisoner 1 would gain from playing Fink. Similarly, prisoner 2 would deviate from playing (Mum, Mum) or (Fink, Mum).

Example 2: The Battle of the Sexes – Games with Multiple NE

Figure 1.4

Chris \ Pat	Opera	Fight
Opera	2, 1	0, 0
Fight	0, 0	1, 2

This interesting game is a coordination game with certain conflict elements. A couple want to spend the evening together, but Pat wants to be together at the prize fight whilst Chris wants to go to the opera together. The game is shown in Figure 1.4.

To solve this game, start with the strategy combination (Fight, Fight).

1. Look at Chris' payoffs. If Pat goes to the fight, is the fight optimal for Chris? Yes, because 1 > 0.
2. Now look at Pat's payoffs. If Chris goes to the fight, is the fight optimal for Pat? Yes, because 2 > 0.

Thus, (Fight, Fight) is an NE. Similarly, (Opera, Opera) is also an NE.

3. Now, consider the strategy combination (Fight, Opera). If Pat goes to the Opera, is Fight optimal for Chris? No, because it gives Chris a payoff of 0, and he can do better by going to the Opera, which would give a payoff of 2.

Thus, (Fight, Opera) is not an NE. Neither is (Opera, Fight).

In conclusion, there are two pure-strategy Nash equilibria in this game, namely (Opera, Opera) and (Fight, Fight). This actually shows a drawback of NE as a solution concept – it does not always provide a unique solution.

Based on Examples 1 and 2, you should now be able to interpret the relation between iterated elimination of strictly dominated strategies and NE:

- If a strategy profile, s^*, is an NE, then it will survive iterated elimination of strictly dominated strategies. Meanwhile, if iterated elimination of strictly dominated strategies eliminates all but s^* then s^* is the unique NE (Prisoners' Dilemma).
- However, there can be strategy profiles that survive strictly iterated elimination of dominated strategies, but they are not NE – for example, (Fight, Opera) and (Opera, Fight) in the Battle of the Sexes.

1.3 Mixed-Strategy Nash Equilibrium

Before we introduce the third example, turn again to your key text by Gibbons (1992) to learn about mixed strategies.

Reading 1.3

Please read Gibbons (1992), Section 1.3.A, pp. 29–33.

Make sure your notes help you to identify the strategies.

Gibbons (1992) Section 1.3.A 'Mixed strategies' in *A Primer in Game Theory*. pp. 29–33.

Example 3: Zero-Sum Games

A zero-sum game is a game of conflict. Any gain for one player comes at the cost of its opponent. Think of tax policy. If the total tax amount is fixed, then the problem is about tax redistribution between people. A simple zero sum game is matching pennies as discussed in Section 1.3.A: a two-player game in which Player 2 gets 1 penny from Player 1 if both pennies match, and loses 1 penny if they don't. The game is illustrated as follows.

Figure 1.5

Player 1 \ Player 2	Heads	Tails
Heads	–1, 1	1, –1
Tails	1, –1	–1, 1

This game has no pure strategy NE because no pure strategy (heads or tails) is a best response to a best response of the other player. At every pure strategy set in this game, both players have an incentive to deviate. In this scenario, what would the players do? To find an equilibrium, a solution is randomising between playing Heads and Tails, and this randomisation is a mixed strategy. Each player could flip a coin and play Heads with probability ½ and Tails with probability ½. In this way, each player makes the other indifferent between choosing Heads or Tails, so neither player has an incentive to deviate. Gibbons has provided a formal definition of mixed strategy in the key text.

Definition 3

Let G be a game with strategy spaces $S_1, S_2, ..., S_I$. A mixed strategy σ_i for player i is a probability distribution on S_i (over the set of pure strategies).

Furthermore, by observing Figures 1.3.1 and 1.3.2 on page 32 of Gibbons, consider the two important concepts illustrated through these examples.

1. A given pure strategy may be strictly dominated by a mixed strategy, even if the pure strategy is not strictly dominated by any other pure strategy.
2. A given pure strategy can be a best response to a mixed strategy, even if the pure strategy is not a best response to any other pure strategy.

A pure strategy is a special case of a mixed strategy, in which the probability distribution over a set of pure strategies for a player assigns a probability equal to one to a single pure strategy and a probability of zero to all the rest.

A strategy is fully mixed, if it assigns to every action a non-zero probability. Now, we go back to the game of Matching Pennies. In a NE, if a player randomises between two different actions, then the player is indifferent between the two actions. This means that the two actions must yield the same expected payoff. (It is very important for you to be able to calculate the probabilities and find the mixed NE.)

Assume that player 1 plays a mixed strategy of Heads with probability *r* (and tails with probability 1 − *r*), and player 2 plays Heads with probability *q*.

Figure 1.6

Player 1 \ Player 2	Heads (q)	Tails ($1-q$)
Heads (r)	−1, 1	1, −1
Tails ($1-r$)	1, −1	−1, 1

Key idea: player 1 must be indifferent between playing Heads and Tails.

Player 1's expected payoff from playing Heads is:

$$q\cdot(-1)+(1-q)\cdot 1=1-2q$$

➔ Player 1's expected payoff from playing Tails is:

$$q\cdot 1+(1-q)\cdot(-1)=2q-1$$

➔ These two expected payoffs must be equal:

$$1-2q=2q-1 \Rightarrow q=\tfrac{1}{2}$$

Now, assuming player 1 randomises, we can work out the expected payoff for player 2. Still, player 2 must be indifferent between playing Heads and Tails.

➔ Player 2's expected payoff from playing Heads is:

$$r\cdot 1+(1-r)(-1)=2r-1$$

➔ Player 2's expected payoff from playing Tails is:

$$r\cdot(-1)+(1-r)\cdot 1=1-2r$$

➔ These two expected payoffs must be equal:

$$2r-1=1-2r \Rightarrow r=\tfrac{1}{2}$$

Review Question 1.1

Thus, the mixed-strategy NE is {½H + ½T, ½H + ½T}. So this gives us two important questions:

1. Are there any other equilibria in this game?
2. If two players can choose any combination, why do they choose these probabilities?

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1. Player 1 loses with probability: $r \cdot q + (1-r) \cdot (1-q) = 1 - q + r(2q-1)$ and wins with probability: $r \cdot (1-q) + (1-r) \cdot q = q + r(1-2q)$

   If $q > \frac{1}{2}$, then $2q - 1 > 0$ ➔ the higher value of q, the lower chance of winning. Then, player 1 would choose Tails, $r = 0$.

   If $q < \frac{1}{2}$, then $2q - 1 < 0$ ➔ the higher value of r, the higher chance of winning. Then, player 1 would choose Heads, $r = 1$. Therefore, there are no other mixed strategy equilibria.
2. It is because these are the probabilities that make the other player indifferent. The probability of ½ is not randomising; ½ is the player's best response to the other player's belief – *ie* the best thing they can do facing uncertainty. Now let us look at the definition of mixed strategy NE.

Definition 4

A mixed-strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ is a Nash equilibrium if and only if

$$u_i\left(\sigma_i^*, \sigma_{-i}^*\right) \geq u_i\left(s_i, \sigma_{-i}^*\right)$$

for all i and $s_i \in S_i$.

Similar to pure strategy NE, mixed strategy NE models a steady state of a game in which players' choices are regulated by probabilistic rules.

Reading 1.4

Please now read Gibbons (1992) 1.3.B, pp. 33–48.

Make sure your notes are sufficient to enable you to revise the important points from them.

Gibbons (1992) Section 1.3.B 'Existence of Nash equilibrium' in *A Primer in Game Theory*. pp. 33–48.

Let us look at the best response *correspondences*. Gibbons has introduced the concept in pp. 42–43. The intuition is that it is the optimal action for a player as a function of the strategies of all other players. If there is always a unique best action given what the other players are doing, this is a function. If for an opponent's strategy, a set of best responses is equally good, it is a *correspondence*.

Recall that player 1's expected payoff from playing Heads ($r = 1$) if player 2 plays q is:

$$q \cdot (-1) + (1-q) \cdot 1 = 1 - 2q$$

And player 1's expected payoff from playing Tails ($1 - r = 1$) if player 2 plays q is:

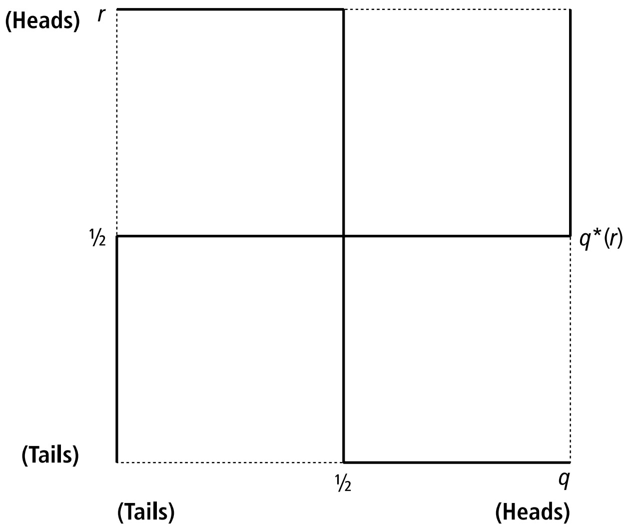
$$q \cdot 1 + (1-q) \cdot (-1) = 2q - 1$$

$$\text{Best response for player 1: } r^*(q) = \begin{cases} 1 & \text{if} \quad q < \frac{1}{2} \\ [0,1] & \text{if} \quad q = \frac{1}{2} \\ 0 & \text{if} \quad q > \frac{1}{2} \end{cases}$$

$$\text{Best response for player 2: } q^*(r) = \begin{cases} 1 & \text{if} \quad r > \frac{1}{2} \\ [0,1] & \text{if} \quad r = \frac{1}{2} \\ 0 & \text{if} \quad r < \frac{1}{2} \end{cases}$$

Best Response Correspondence below shows that both correspondences intersect at only one point at which $r = q = ½$. This gives us the unique mixed strategy NE of this game.

Figure 1.7



We now go back to the Battle of Sexes. We have found the pure-strategy NEs, so how about the mixed-strategy NEs?

Figure 1.4

Chris \ Pat	Opera	Fight
Opera	2, 1	0, 0
Fight	0, 0	1, 2

Let $(r, 1 - r)$ be the mixed strategy in which Chris plays Opera with probability r. Thus, you can analyse the game with the following steps.

If Pat plays $(q, 1 - q)$, Chris's expected payoff from Opera is:

$$2q + 0(1 - q) = 2q$$

And Chris's expected payoff from Fight is: $0q + 1(1 - q) = 1 - q$

➔ Chris goes to Opera if $2q > 1 - q$ ➔ $q > 1/3$ (*ie* $r = 1$)

And Chris goes to Prize Fight if $q < 1/3$ (*ie* $r = 0$)

Thus, if $q = 1/3$, any value of r is a best response.

Similarly, if Chris plays $(r, 1- r)$, Pat's expected payoff from Opera is:

$$r + (1 - r)\,0 = r$$

And Pat's expected payoff from Fight is:

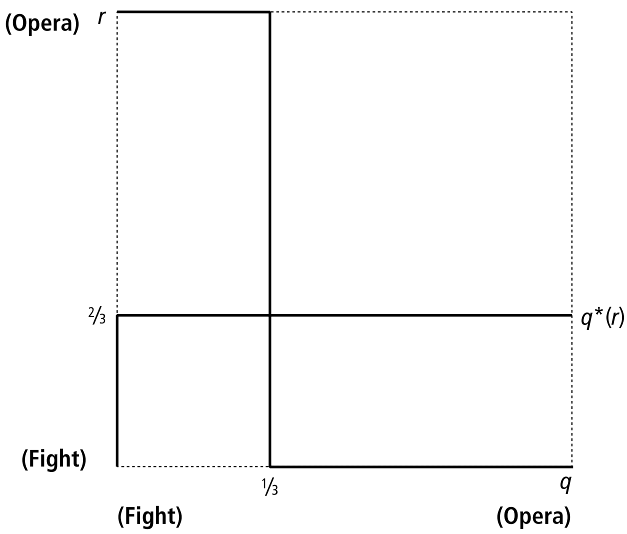
$$0r + 2(1 - r) = 2(1 - r)$$

➔ Pat goes to Opera if $2r > 2(1 - r)$ ➔ $r > 2/3$ (*ie* $r = 1$)

And Pat goes to Prize Fight if $r < 2/3$ (*ie* $r = 0$)

Thus, if $r = 2/3$, any value of q is a best response.

Figure 1.8



By drawing the best response function above, you can see that both correspondences intersect at three points. Hence, there are three NEs in this game: {Opera, Opera}, {Fight, Fight} and {⅔ Opera + ⅓ Fight, ⅓ Opera + ⅔ Fight}.

So which strategy is better, pure or mixed, in the Battle of the Sexes?

To compare these strategies, let us compare Chris's payoff in the mixed equilibrium.

$$rq(2) + (1-r)q(0) + r(1-q)(0) + (1-r)(1-q)(1)$$
$$= \frac{2}{3}\cdot\frac{1}{3}\cdot 2 + 0 + 0 + \frac{1}{3}\cdot\frac{2}{3}\cdot 1 = \frac{2}{3}$$

This is the same as Pat's payoff. Indeed, both of them are worse off in the mixed-strategy NE. Hence, one may think that the two players would want to avoid this equilibrium.

1.4 Existence of Nash Equilibrium

By finding the mixed-strategy Nash equilibria in the above examples, we can now introduce Nash's Existence Theorem. The theorem shows that given a game with a finite number of strategies for each player, there is at least one (mixed-strategy) Nash equilibrium. As you saw in your last reading, the proof is based on Kakutani's fixed-point theorem, which is a generalisation of Brouwer's fixed-point theorem. You should know the intuition of this theorem, but you do not need to prove the theorem. This theorem shows that NE is 'stronger' than Iterated Deletion of Dominated Strategies – *ie* there is at least one solution to every finite strategic-form game. Furthermore, a NE cannot be strictly dominated, although it may be weakly dominated. To achieve the outcome of NE, it requires not only rationality but also common beliefs or expectations of what will happen in the game.

1.5 Applications of Nash Equilibrium

The theory of Nash equilibrium is a great tool to clarify the structure and equilibrium of duopoly and oligopoly markets. For instance, you can figure out how each firm reacts to its rival's strategy, what the market equilibrium is, *etc*. The concepts of oligopoly date back to Cournot (1838) and Bertrand (1883). Cournot was the first to investigate non-cooperative competition between two producers – that is, the so-called duopoly problem. He assumes that two firms produce the same product and the price of the product depends on the total quantity produced. Later on, Bertrand analysed price competition between two producers. He finds that each firm charges the price at the marginal cost in equilibrium, and it is the same under perfect competition. In this section, we apply the concepts of NE to both models.

In a market, an important question for firms is how to choose between 'price' and 'quantity' as the decision variables. The particular choice depends on the specific situations of the industry. Kreps and Scheinkman (1983) adopt a two-stage game to explain the differences between the Bertrand and Cournot models. Usually firms first make long-run decisions by choosing their capacities and then decide their short-run prices. Your next reading considers these applications, and you should be able to analyse the following examples yourself.

Reading 1.5

Please now read Gibbons Sections 1.2.A–1.2.C, pp. 14–26. Pay particular attention to Sections 1.2.A 'Cournot Model of Duopoly' and 1.2.B 'Bertrand Model of Duopoly'.

Gibbons (1992) Sections 1.2.A 'Cournot model of duopoly' to 1.2.C 'Final-offer arbitration' in *A Primer in Game Theory*. pp. 14–26.

1.5.1 Cournot model of duopoly

As you read above, you start with this:

- Players: two firms
- Strategy: a set of possible outputs (any nonnegative amount)
- Payoff: profit of each firm.

In a one-shot simultaneous game, firm *i* chooses its output level, q_i, and has no fixed cost, but a unit cost $C_i(q_i) = cq_i$ where $c < a$. The products are assumed to be homogenous (*ie* they are perfect substitutes), so the market demand, $p(q_1 + q_2)$, determines the price. *P* is an inverse demand function: $P(Q) = a - Q$, where $Q = q_1 + q_2$. Therefore, the payoff function can be derived as follows:

$$\pi_i(q_i, q_{-i}) = q_i(P(Q) - c) = q_i\left[a - (q_1 + q_2) - c\right]$$

To solve NE, each firm *i* needs to choose q_i to maximise its profit function. Thus, for firm 1,

$$\max_{0 \le q_i < \infty} \pi_1(q_1, q_2^*) = \max_{0 \le q_i < \infty} q_1\left[a - (q_1 + q_2^*) - c\right]$$

For each firm, first-order conditions (FOCs) with respect to q_1 and q_2 give the best response functions:

$$B_1(q_2^*) = q_1^* = \frac{1}{2}(a - q_2^* - c)$$

$$B_2(q_1^*) = q_2^* = \frac{1}{2}(a - q_1^* - c)$$

Here, given that $\frac{dB_i}{dq_{-i}} < 0$, the best responses are called *strategic substitutes* – that is, two products mutually offset one another. An increase in one firm's output decreases the marginal revenues of the others.

Now solving the system of equations gives

$$q_1^* = q_2^* = \frac{a - c}{3}$$

and $Q^* = \frac{2(a-c)}{3}$. Substituting Q^* into $P(Q) = a - Q$ gives the NE price at

$$P^* = \frac{a + 2c}{3}$$

This game can be extended into *n*-firms as $p \to c$ and $n \to \infty$.

Alternatively, this two-firm game is dominance-solvable, as demonstrated below. This means that iterated elimination of dominated strategies picks out the unique NE.

Suppose that the firm produces q_m, then its profit would be

$$\pi_i\left(q_m, q_{-i}\right) = \frac{a-c}{2}\left(a - \left(\frac{a-c}{2} + q_{-i}\right) - c\right) = \frac{a-c}{2}\left(\frac{a-c}{2} - q_{-i}\right)$$

Now suppose that the firm chooses to produce $q_m + x$, with $x > 0$, then the profit will be

$$\pi_i\left(q_m + x, q_{-i}\right) = \left(\frac{a-c}{2} + x\right)\left(\frac{a-c}{2} - x - q_{-i}\right) = \pi_i\left(q_m, q_{-i}\right) - x\left(x + q_{-i}\right)$$

Hence, the monopoly quantity $q_m = \frac{a-c}{2}$ strictly dominates any higher quantity. Given that $q_i > q_m$ is eliminated, then any $q_i < B_i\left(q_m\right) = \frac{a-c}{4}$ is dominated by $\frac{a-c}{4}$. To see it, note

$$\pi_i\left(\frac{a-c}{4}, q_{-i}\right) = \frac{a-c}{4}\left(\frac{3(a-c)}{4} - q_{-i}\right).$$

Thus,

$$\pi_i\left(\frac{a-c}{4} - x, q_{-i}\right) = \left(\frac{a-c}{4} - x\right)\left(\frac{3(a-c)}{4} + x - q_{-i}\right) = \pi_i\left(q_m, q_{-i}\right) - x\left(\frac{a-c}{4} + x - q_{-i}\right)$$

After these two rounds of elimination, the quantities remaining in each firm i are limited to $\left[\frac{a-c}{4}, \frac{a-c}{2}\right]$. Repeating these arguments leads to ever-smaller intervals. In the limit these intervals converge to the unique NE of $q_i^* = \frac{a-c}{3}$.

However, the more-than-two-firm cases are not dominance-solvable. As explained above, using dominance to solve a game requires us to delete dominated strategies for each of the players; then to solve the smaller game using the same process until there is no further possible elimination. The game is dominance-solvable, if only single strategies remain.

1.5.2 Bertrand model of duopoly

Let us now consider the Bertrand model. Gibbons gives the details of heterogeneous products, but it is necessary for you to consider the case of homogenous products first. You should be able to understand the intuitions as well as derive the cases.

- Player: two firms
- Strategy: a set of possible prices (any nonnegative amount)
- Payoff: profit of each firm.

Case 1: Homogenous Products

In a one-shot simultaneous game, firm *i* chooses its price, p_i, and has no fixed cost, but a symmetric unit cost *c*. The demand for firm *i* is

$$q_i\left(p_i, p_{-i}\right)=\begin{cases} D(p_i) & \text{if} \quad p_i < p_{-i} \\ \dfrac{D(p_i)}{2} & \text{if} \quad p_i = p_{-i} \\ 0 & \text{if} \quad p_i > p_{-i} \end{cases}$$

$$\pi_i\left(p_i, p_{-i}\right)=\begin{cases} (p_i-c)D(p_i) & \text{if} \quad p_i < p_{-i} \\ \dfrac{(p_i-c)D(p_i)}{2} & \text{if} \quad p_i = p_{-i} \\ 0 & \text{if} \quad p_i > p_{-i} \end{cases}$$

where $D(p)$ is the market demand. Therefore, when two firms have $c_1 = c_2 = c$, the unique NE is to set $p_1 = p_2 = c$ in which each firm gets half of the market.

To understand that, let us consider this: if $p_1 > c$, firm 2 sets $p_2 = p_1 - \varepsilon$ and gets the whole market. If $p_2 > c$, firm 1 has the incentive to undercut the price further, so this cannot be an equilibrium. The intuition is that if a firm charges a lower price than its rival, then the firm will obtain the whole demand. If two firms charge the same price, the market demand is split equally.

Therefore, if all the firms are identical, $c_i = c$, for all *i*, then $p = c$ for any $n \geq 2$, market price equals the marginal cost, and thus firms make zero profits. This leads to the Bertrand paradox: having two firms in the industry is enough to obtain perfect competition (rather than achieve monopoly outcomes). This seems implausible.

Case 2: Heterogeneous Products

The demand for firm *i* is

$$q_i\left(p_i, p_{-i}\right) = a - p_i + bp_{-i}$$

where $b > 0$, so firm *i*'s product is a substitute for firm –*i*'s product. And firm *i*'s profit is

$$\pi_i\left(p_i, p_{-i}\right) = q_i\left(p_i, p_{-i}\right)\left(p_i - c\right) = \left(a - p_i + bp_{-i}\right)\left(p_i - c\right)$$

To solve the NE, each firm maximises its profit. For firm 1,

$$\max_{0 \leq p_i < \infty} \pi_1(p_1, p_2^*) = \max_{0 \leq p_i < \infty} \left(a - p_1 + bp_2\right)\left(p_1 - c\right)$$

FOC gives the best response function,

$$B_1\left(p_2\right) = \frac{1}{2}\left(a + bp_2^* + c\right)$$

Symmetrically,

$$B_2(p_1) = \frac{1}{2}\left(a + bp_1^* + c\right)$$

Solving the above two equations gives the unique NE at

$$p_1^* = p_2^* = \frac{a+c}{2-b}$$

Differentiating the product allows the firms to get out of the destructive logic of Bertrand price competition under the homogenous product case. Therefore, it is possible to achieve the equilibrium prices at a higher level than under perfect competition.

1.5.3 Final offer arbitration

Final offer arbitration by a third party has become a popular method of conflict resolution in many areas. It is used in the settlement of disputes under existing contracts, typically when the unions (*eg* police or some public services) are prohibited from striking. The intuition is stated in Gibbons 1.2.C. You are not required to derive the equilibrium. In final offer arbitration, each party submits a final proposal to an arbitrator. The arbitrator chooses one of them. Different from conventional arbitration, the arbitrator is not allowed to compromise the demands of the parties. The intuition behind the equilibrium is that each party has a trade-off. A more demanding offer yields a better payoff, but it will be less likely to be chosen by the arbitrator. If there is little uncertainty about the arbitrator's preferred settlement, the parties are likely to make offers close to the mean because the arbitrator is very likely to choose the settlement close to the mean. As the uncertainty increases, the parties' offers become more demanding. This mechanism provides an incentive for the parties to reach a negotiated settlement.

1.5.4 The problem of the commons

In 1968, Hardin wrote an influential paper, named 'The tragedy of the commons', in which he explains a crucial problem on the exploitation of the natural resources and environment. Without any automatic mechanism or incentive, resources are over-exploited because agents maximise their own benefits as a result of their selfish behaviour. The problem is the difference between public and private incentives. Indeed, this issue has become more and more important nowadays. It happens wherever there is a resource open to everyone, such as a hunting area, grazing land, *etc*.

We first consider a simple example that is not mentioned in Gibbons. Consider n farmers in a village with limited grassland and each farmer has the option to keep either one sheep or no sheep. The utility of a sheep from producing wool is 1, and the pollution to the environment from a sheep is 5. Let X_i be a variable, either 0 or 1.

$$X_i = \begin{cases} 1 & i^{\text{th}} \text{ farmer keeps a sheep} \\ 0 & \textit{otherwise} \end{cases}$$

Given the pollution is shared among all farmers, the utility of i^{th} farmer, $U_i(X_i)$, is

$$U_i(X_i) = X_i - \frac{5(X_1 + X_2 + ... + X_n)}{N}$$

Could you derive the NE for all i if $N \geq 5$? This is not essential, but you should understand the reasoning behind it. The NE is $X_i = 1$, because keeping a sheep would add more utility to a farmer from wool than it would subtract utility from them due to pollution. Therefore, each farmer keeps a sheep and the utility for every farmer is −4.

Consequently, this can cause excessive pollution, and the tax is likely to be imposed. But how much tax would that be? The tax should be equal to the total damage done to the environment. Therefore, the utility of i^{th} farmer becomes

$$U_i(X_i) = X_i - 5X_i - \frac{5(X_1 + X_2 + ... + X_n)}{N}$$

In this case, NE is $X_i = 0$ for all i, since the farmer increases his utility by giving away his sheep.

Reading 1.6

Please now read Gibbons Sections 1.2.D, pp. 27–29, for another example of tragedy of commons. Again, you are not required to derive the NE, but you should understand the intuitions. You should pay particular attention to the equations (the first-order conditions) below, which relate to the NE

$$v(G^*) + \frac{1}{n}G^* v'(G^*) - c = 0$$

and the social optimum

$$v(G^{**}) + G^{**}v'(G^{**}) - c = 0$$

Gibbons (1992) Section 1.2.D 'The problem of the commons' in *A Primer in Game Theory*. pp. 27–29.

Here, the value to a farmer of grazing a goat on the green when a total of G goats are grazing is $v(G)$ per goat. Intuitively, $G^{**} < G^*$ and the difference between NE and social optimum is due to $\frac{G^*}{n}v_i'(G^*)$ and $G^{**}v'(G^{**})$. These externalities cannot be easily internalised. Therefore, the problem of the commons shows the inefficiency of NE.

1.6 Conclusion

Unit 1 has provided an introduction to game theory. It began with the static games of complete information. You have studied the basic concepts, such as pure- and mixed-strategy Nash equilibria. You have also studied the sufficient conditions and intuitions of the existence of Nash equilibrium. You have applied NE to simple oligopoly games (Cournot & Bertrand), and evaluated the intuition of final-offer arbitration and the problem of the commons. To check your understanding of this, look again at the Learning Outcomes at the beginning of the unit and make sure you can achieve them.

In the next unit, you will look at another type of game – dynamic games of complete information. That will allow you to apply the concept to more complicated market situations.

Optional Reading 1.1

If you would like to explore the ideas behind the Nash equilibrium you might like to read his article:

Nash JF (1950) 'Equilibrium points in N-person games'. *Proceedings of the National Academy of Sciences*, 36, 48–49.

1.7 Exercises

Exercise 1.1

Solve Gibbons' Problem 1.2 on page 48.

Exercise 1.2

Consider the n-player Cournot model. Each firm chooses its own quantity to produce, $q_i \geq 0$, simultaneously. The price for each unit is given by $p = 1 - Q$ where

$Q = \sum_{i=1}^{n} q_i$.

Assuming no costs, the profit of each firm is $\pi_i = pq_i$.

a) Find the NE and show that it is unique and symmetric.
b) Suppose that $n = 2$. Which strategies for each player survive iterated deletion of strictly dominated strategies?
c) Now suppose that $n = 3$. Reconsider part (b).

Exercise 1.3

Show that the unique profile of strategies that survives iterated removal of strictly dominated strategies is a unique Nash Equilibrium.

Exercise 1.4

There are two firms in an industry. Let q_1 and q_2 be the output of Firm 1 and Firm 2 and $Q = q_1 + q_2$ be total output. The inverse demand in the industry is $P(Q) = 45 - Q$. The cost function for each firm is
$C(q_i) = 9q_i$.

a) Assume a two-firm Cournot model. Calculate Firm 1's best response function $q_1 = R(q_2)$ to the output choice of Firm 2, and Firm 2's best response function $q_2 = R(q_1)$ to the output choice of Firm 1.
b) Compute the equilibrium price, quantities, (P, q_1, q_2) and each firm's profit level.
c) Assume a two-firm Bertrand model. Solve for the equilibrium prices and outputs of Firms 1 and 2, assuming that consumers will split evenly between the two firms, if the firms offer the same price. Compute each firm's profit level.

Exercise 1.5

Given that $u > w$, $y > m$, $n > v$, and $x > z$, consider the following normal form game, where player 1 chooses rows and player 2 chooses columns.

Figure 1.9

Player 2

Player 1	Left	Right
Up	*u*, *v*	*m*, *n*
Down	*w*, *x*	*y*, *z*

a) Prove that this game has no pure strategy Nash Equilibrium.
b) Solve for the mixed strategy Nash Equilibrium in terms of the parameters.

✍ Exercise 1.6

Solve Gibbons' Problem 1.12, p. 51.

1.8 Answers to Exercises

Exercise 1.1

For player 1, B is strictly dominated by T ➔ Eliminate the row of B.

Then for player 2, C is strictly dominated by R. Thus, the games can be reduced to a 2 × 2 game. The strategies that survive IESDS are: T and M for 1, and L and R for 2.

The pure strategy Nash Equilibria of the game are (M, L) and (T, R).

Exercise 1.2

a) Each player i is to maximise its profit, by solving

$$\max \pi_i = pq_i$$

$$\pi_i = (1-Q)q_i = (1-q_i-q_{-i})q_i$$

FOC with respect to q_i implies

$$\frac{d\pi_i}{dq_i} = 1-2q_i-q_{-i} = 0$$

$$q_i = \frac{1-q_{-i}}{2} \quad \text{or} \quad q_i = 1-Q$$

Since this condition is the same for all i, the equilibrium must be symmetric. Solving, we obtain

$$q_i = \frac{1}{1+n};\ Q = \frac{n}{1+n};\ p = \frac{1}{1+n};\ \pi_i = \frac{1}{(1+n)^2}$$

b) Since $q_2 \geq 0$, FOC above gives us $q_1 \leq 1/2$ (all other strategies are strictly dominated by $q_1 = 1/2$). Since $q_1 \leq 1/2$, it gives $q_2 \geq 1/4$; and this gives $q_1 \leq 3/8$, so $q_2 \geq 5/16$. Repeated iterations and symmetry give $q_1 = q_2 = 1/3$.

c) Since $q_2 \geq 0$, FOC above gives us $q_1 \leq 1/2$ (all other strategies are strictly dominated by $q_1 = 1/2$). Since $q_1 \leq 1/2$ and $q_3 \leq 1/2$ it gives $q_2 \geq 0$. Iterated deletion of strictly dominated strategies implies only that $0 \leq q_1, q_2, q_3 \leq 1/2$.

Exercise 1.3

This proof can be done by explaining the intuition of Iterated Deletion of Dominated Strategies and Nash Equilibrium.

First, if a pure strategy $s_i \in S_i$ for player i is strictly dominated by another pure strategy $s_i' \in S_i$ then any mixed strategy that has positive probability to s_i is strictly dominated by some other mixed strategy. If a pure strategy is not strictly dominated by another pure strategy, it is also not strictly dominated by any mixed strategy.

Second, if a mixed strategy μ_i for i has positive probability only to pure strategies that are not strictly dominated then μ_i, then it is not strictly dominated by any other mixed strategy.

Third, if each player $i = 1, i \ldots, N-1$ has a strictly dominant strategy s_i (s_i strictly dominates every other strategy), then the game is dominance-solvable.

There is a unique solution to iterative deletion of strictly dominated strategy.

Exercise 1.4

a) Firm *i* maximises its profit taking the Firm $-i$'s choice, q_{-i} as given.

$$\max_{q_i} \left(45 - q_i - q_{-i}\right) q_i - 9q_i$$

Set FOC equal to zero,

$$\frac{d\pi_i}{dq_i} = 45 - 2q_i - q_{-i} - 9 = 0$$

Solving this gives the best response function:

$$q_i = \frac{36 - q_{-i}}{2} \Rightarrow q_1 = \frac{36 - q_2}{2} \text{ and } q_2 = \frac{36 - q_1}{2}$$

Solving these 2 equations gives: $q_1 = q_2 = 12$. The equilibrium price is 45 – 12 – 12 = 21.

Therefore, the profit of each firm will be

$$\pi_1 = \pi_2 = \left(45 - 24\right) \times 12 - 9 \times 12 = 144$$

b) In price competition, firms always want to undercut their rivals slightly to capture the whole market, as long as the rival's price is above the marginal cost. Therefore, the only equilibrium under Bertrand competition is for the two firms charging $P_1 = P_2 = MC = 9$. In such a case, no firm has an incentive to deviate, because any price above the marginal cost will not sell while any price lower than the marginal cost incurs a loss. Consequently, $Q = 45 - P = 45 - 9 = 36$. Symmetrically, $q_1 = q_2 = 18$. This leads to $\pi_1 = \pi_2 = 0$. This shows that the market outcome under Bertrand competition is identical to the perfectly competitive market outcome.

Exercise 1.5

First, state the definition of NE.

There is no pure strategy NE in this game.

Under (U, R), player 1 will deviate to D.

Under (D, R), player 2 will deviate to L.

Under (U, L), player 2 will deviate to R.

Under (D, L), player 1 will deviate to U.

To find the mixed strategy NE, suppose player 1 plays U with probability p and player 2 plays L with probability q. Therefore,

The expected payoff for player 1:

$$EU_1\left(U\right) = qu + \left(1-q\right)m = qw + \left(1-q\right)y = EU_1\left(D\right)$$

➔ $q = \frac{y-m}{(y-m)+(u-w)}$

The expected payoff for player 2:

$$EU_2(L) = pv + (1-p)x = pn + (1-p)z = EU_2(R)$$

➔ $p = \frac{x-z}{(x-z)+(n-v)}$

✍ **Exercise 1.6**

Let p denote the probability that player 1 plays T, and q denote the probability that player 2 plays L. In a mixed strategy equilibrium, the following two conditions have to be satisfied:

i) Given q, player 1 is indifferent between playing T and B. Therefore,

$$2q = q + 3(1-q)$$

➔ $q = 3/4$

ii) Given p, player 2 is indifferent between playing L and R. Therefore

$$p + 2(1-p) = 2p$$

➔ $p = 2/3$

The mixed strategy NE is $\left(\frac{2}{3}T + \frac{1}{3}B\right)$ and $\left(\frac{3}{4}L + \frac{1}{4}R\right)$.

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